A Matched Delay Approach to Subtractive Linear Phase High-Pass Filtering

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I. Introduction

When removing additive noise outside the frequency range of a desired signal by high-pass or low-pass filtering, a linear phase response (a phase shift proportional to frequency) within the passband best preserves the waveform of the signal, except for a time delay proportional to the slope of the phase response. Though techniques for linear phase low-pass filtering are well known, analog methods for constructing linear phase high-pass filters are not generally discussed in the standard references on filter design, except for the case in which the passband is small compared to the high-pass cutoff frequency (i.e., the bandpass case). This paper presents a semianalog method for linear phase high-pass filtering that does not have this restriction.

To reduce the phase distortion in an analog high-pass filter, it would be necessary to have a phase shift at the highest frequency of interest in the passband f_H that is larger than the phase shift at the low frequency limit f_L by the ratio f_H/f_L. The phase shift at the band edge f_L typically ranges from about $\pi/4$ radians in the simplest filter to five or ten times that figure in more complex and powerful filters. If $f_H \gg f_L$, the required high frequency phase shift is very large. To obtain this phase shift by standard analog methods, i.e., with a circuit having a rational Laplace transform, it is necessary to have an impractically large number of poles and zeros, since the phase shift due to each pole or zero is limited to 90°. (The same relation between f_H and f_L holds for a low-pass filter, but there is no resulting large phase shift required, since the largest phase shift in the passband would occur at the band edge f_H, and have a magnitude similar to that of f_L in the high-pass case.)

The linear phase high-pass filter presented here consists of two forward paths as shown in Fig. 1. The high-pass characteristic is obtained by subtracting the output of a low-pass forward path (H_2 in the figure) from the output of an all-pass network (H_1). We shall refer to this type of system as a "subtractive" high-pass filter.

The case in which $H_1 = 1$ is shown by [1] to result in a high-pass filter with little overshoot and an asymptotic rejection of 20 dB/decade for any rational transform low-pass filter H_2, independent of the order of H_2. However, since this type of subtractive high-pass filter still has a rational transform it cannot approach a linear phase response without having an extremely high number of poles.

In this paper, we discuss the case in which H_2 is a low-pass filter having a relatively linear phase (constant delay) characteristic, and H_1 a delay element matched to the delay of H_2. We will call this case a "matched-delay subtractive" (MDS) high-pass filter. If H_1 is an ideal delay, with frequency response $\epsilon^{-j\beta\omega}$, and H_2 a perfectly linear phase low-pass filter, with a frequency response $|H_2(j\omega)|\epsilon^{-j\beta\omega}$, then both H_1 and H_2 would have a delay equal to β, and the overall frequency response would be

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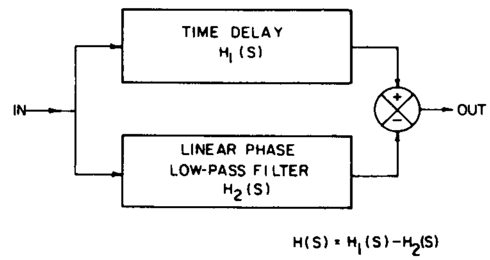

Fig. 1. Block diagram of a matched delay subtractive (MDS) high-pass filter.

$[1-|H_2(j\omega)|]\epsilon^{-j\beta\omega}$. The magnitude factor $[1-|H_2(j\omega)|]$ is the amplitude response and is clearly that of a high-pass filter, while the phase factor $\epsilon^{-j\beta\omega}$ signifies a linear-phase response.

II. MDS Filter Using a Bessel Low-Pass Element

Consider the case in which H_1 is a perfect delay, and H_2 is a realizable low-pass filter with a rational transfer function. To achieve a uniformly good rejection in the stopband of the high-pass filter $H = H_1 - H_2$, the phase response of the low-pass filter H_2 in its passband must closely approximate that of the delay H_1, and so be of nearly linear phase. For H to have a linear phase response in its passband, the low-pass filter H_2 must have a linear phase response near its cutoff frequency. (The phase response at frequencies far into the low-pass stopband is not as important, since the high-pass phase response in this region is determined primarily by the delay element H_1.) If H_2 is to be realized by an all-pole network of order N, a maximally linear-phase approximation occurs with a Bessel (or Bessel/Thomson) low-pass filter [2]. For large N, the magnitude of the transfer function of a Bessel filter becomes increasingly Gaussian, and the amplitude response of the entire filter is therefore approximately $1-\epsilon^{-\alpha\omega^2}$ with α a function of the cutoff frequency. Expanding $\epsilon^{-\alpha\omega^2}$ about $\omega = 0$, we find that for small ω, the amplitude response approaches $1-(1-\alpha\omega^2) = \alpha\omega^2$. An amplitude response asymptotically proportional to ω^2 means that the slope of the amplitude response is asymptotic to 40 dB/decade. The −3-dB frequency f_c of the filter occurs when $1-\epsilon^{-\alpha\omega^2} = 1/2$, and is related to α by

$$f_c = \frac{1}{2\pi}\sqrt{-\frac{1}{\alpha}\ln\left(1-\frac{1}{2}\right)} = \sqrt{\frac{0.0311}{\alpha}}.$$

We might then expect that if H_2 is approximated with a Bessel filter of high enough order, the response of the filter will become relatively independent of the order of the approximation, and approach a rejection of 40 dB/decade for a range of frequencies that increases with the order. This is illustrated in Fig. 2, in which the frequency response of an MDS linear-phase high-pass filter is computed for H_2 approximated by Bessel filters of increasing order. The phase error shown in the figure is the deviation from the delay of H_1. All filters were normalized for the same asymptotic rejection at low frequencies, with the delay H_1 matched to the Bessel low frequency phase response. The frequency normalization is arbitrarily set to make the cutoff frequencies of the various filters somewhere near $\omega = 1$. On this scale, the amplitude response with a Gaussian filter would be nearly indistinguishable from the response with the sixth-order Bessel filter. It can be seen from the figure than for orders greater than about two, the primary effect of increasing the order of H_2 is to reduce the phase error near the cutoff frequency, with amplitude response changing very little. To facilitate a comparison with more conventional

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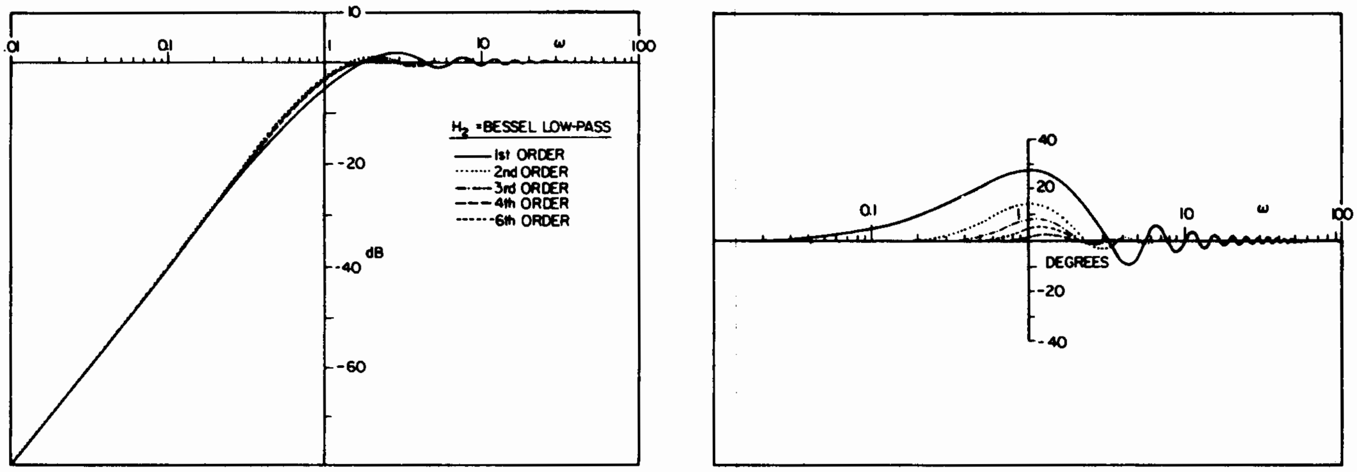
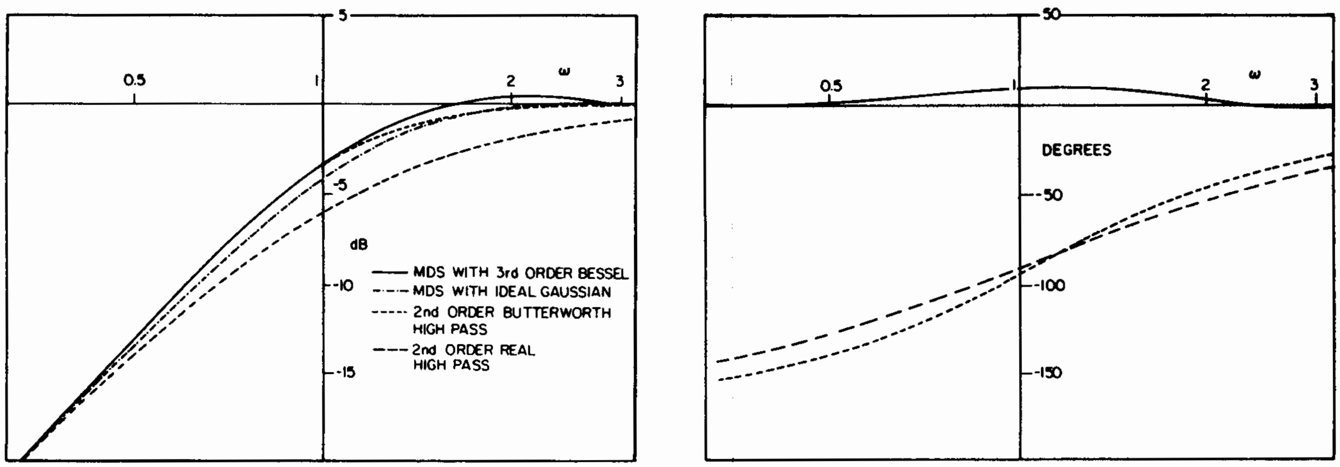


(a) (b)

Fig. 2. Amplitude response and phase error for an MDS filter having a Bessel low-pass element of the order shown in the figure.



(a) (b)

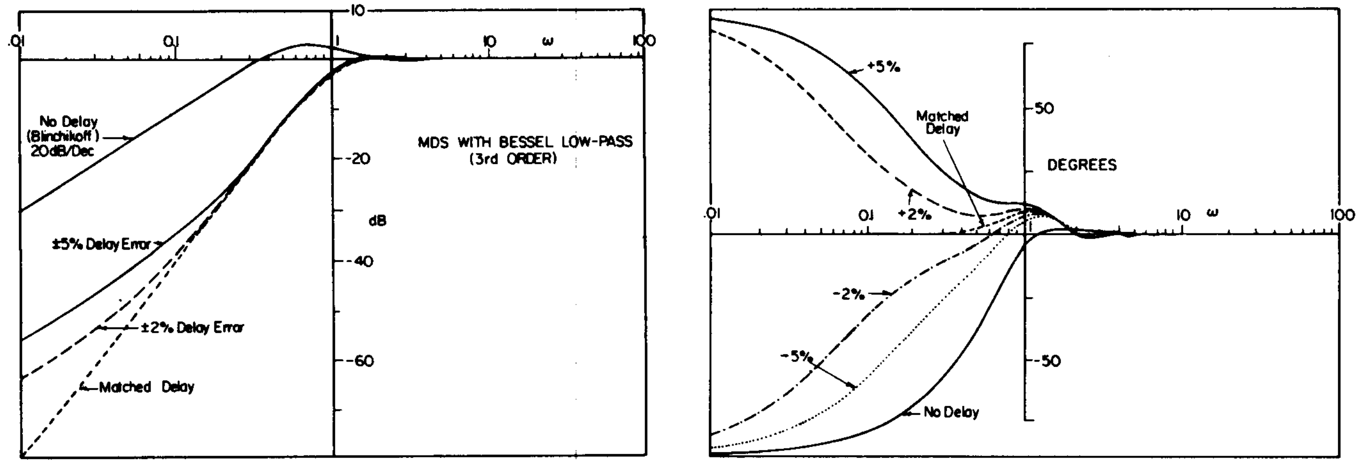
Fig. 3. Amplitude response and phase error for a number of high-pass filter configurations. 1) MDS filter with a third-order Bessel low-pass element. 2) MDS filter with an ideal Gaussian low-pass element. 3) Second-order Butterworth high-pass filter. 4) Second-order high-pass filter having two identical real poles and two zeros at the origin.

rational-transform filters, Fig. 3 compares the amplitude response of two Bessel-derived MDS filters (a third order Bessel and an ideal Gaussian) with a second-order Butterworth high-pass filter and a high-pass filter with two identical real poles. All filters were designed for the same asymptotic rejection at low frequencies. It can be seen in Fig. 3 that the amplitude responses of the MDS filters are similar to that of the two-pole Butterworth filter, and significantly better than that of the double real-pole filter. The phase responses of the standard filters are obviously much worse than those of the MDS filters.

The performance of an MDS filter depends on the matching of the delays of H_1 and H_2. The affect of a gain mismatch would primarily affect the maximum attainable attenuation, and therefore can be easily trimmed by minimizing the output of the filter at low frequencies. A gain mismatch of 1 percent would result in a maximum attenuation of 40 dB.

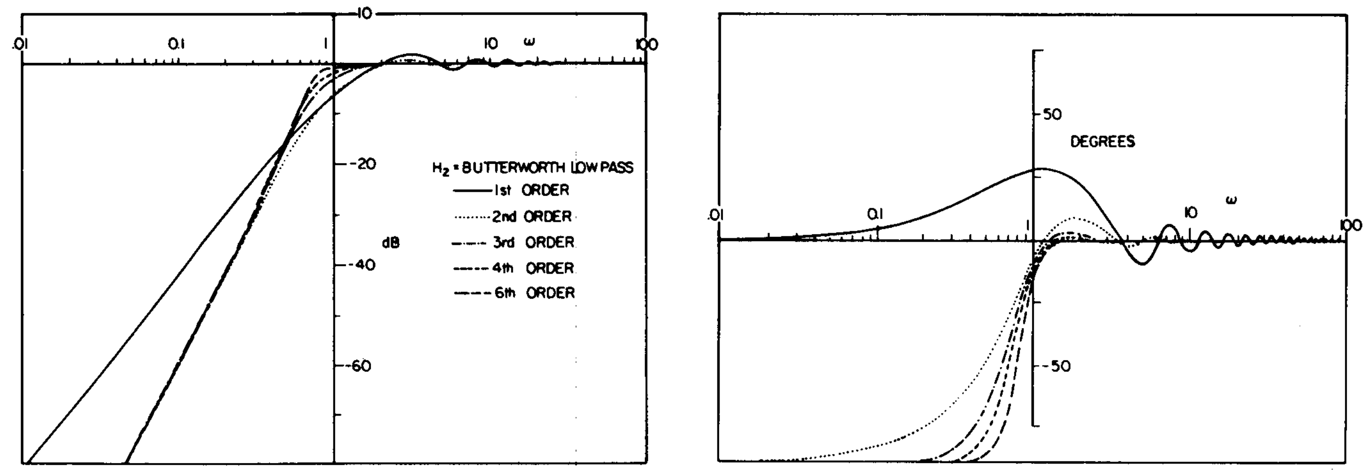
To illustrate the affect of a delay mismatch, Fig. 4 shows the computed frequency response of a third-order Bessel MDS filter with the delays mismatched by ± 2 and ± 5 percent. Also included for comparison is the case of no delay, which corresponds to the subtractive filter proposed by Blinchikoff. As proven by Blinchikoff, the zero-delay filter has an asymptotic rejection of 20 dB/decade, as compared to the 40 dB/decade we have shown holds for the matched-delay case. With a delay error of 5 percent, the amplitude response follows the matched delay response within 1 or 2 dB down to a rejection of about 20 dB, at which point the attenuation begins to decrease to 20 dB/decade. With a delay error of 2 percent, the amplitude response is not affected significantly down to a rejection of about 40 dB. Thus both the delay and the gain of an MDS filter could be trimmed by minimizing the output at low frequencies, but to trim the delay accurately it would be necessary to use the lowest possible frequency. For example, to match delays to within 1 percent it would be necessary to use a frequency at least a decade below the cutoff frequency.

The maximum phase error within the passband also changed very little for delay errors of as large as 2 percent. Since the phase response in the passband stays relatively unchanged, the degrada-



(a) (b)

Fig. 4. Amplitude response and phase error for a third-order Bessel-derived MDS high-pass filter with variation of the delay H_1.



(a) (b)

Fig. 5. Amplitude response and phase error for an MDS filter having a Butterworth low-pass element of the order shown.

tion of signal waveforms within the passband caused by phase error is increased very little by small errors in delay matching.

III. MDS Filter Using a Butterworth Low-Pass Element

Though their maximally linear phase response make the Bessel filters a good choice for the low-pass section of an MDS filter, we have shown that their gradual cutoff characteristic limits the high-pass response to 40 dB/decade. In some applications, it may be desirable to sacrifice some degree of linearity in the phase response of the high-pass filter if a sharper cutoff than 40 dB/decade can be attained. We will show that such a tradeoff can be accomplished using a Butterworth low-pass filter, since the low frequency response of a Butterworth-derived MDS filter falls off asymptotically at 60 dB/decade, provided the order of the Butterworth low-pass filter used is at least two. It will also be shown that the resulting phase distortion (deviation from linearity) of the response in a Butterworth-derived MDS filter occurs primarily near or below the cutoff frequency, and may be acceptable for many applications.

If

$$H(j\omega)=H_1(j\omega)-H_2(j\omega)$$

as in Fig. 1, then from vector algebra it follows that

$$|H|^2=|H_1|^2+|H_2|^2-2|H_1|\,|H_2|\cos(\phi_1-\phi_2)$$

where ϕ_1 and ϕ_2 are the phase angles of H_1 and H_2, respectively, with all H and ϕ functions of ω.

If H_1 is a Butterworth low-pass filter of order N, and H_2 a perfect delay matched to ϕ_1 at low frequencies, then [3]

$$|H_1|^2=\frac{1}{1+\omega^{2N}}$$

$$\phi_1=-\sum_{M=0}^{\infty}\frac{\omega^{2M+1}}{(2M+1)\sin\frac{(2M+1)\pi}{2N}}$$

$$|H_2|^2=1$$

$$\phi_2=\lim_{\omega\to 0}\phi_1=-\frac{\omega}{\sin\frac{\pi}{2N}}.$$

In forming $(\phi_1-\phi_2)$, the subtraction of ϕ_2 removes the first term from the summation for ϕ_1, and $|H|^2$ is then given by

$$|H|^2=\frac{1}{1+\omega^{2N}}+1-2\sqrt{\frac{1}{1+\omega^{2N}}}\cos\left\{\sum_{M=1}^{\infty}\frac{\omega^{2M+1}}{(2M+1)\sin\frac{(2M+1)\pi}{2N}}\right\}.$$

As $\omega\to 0$, the argument of the cosine function approaches the first term in the summation, since the power of ω is higher in all other terms. In addition, we can approximate the cosine function with the first two terms in the expansion

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.$$

This yields

$$\begin{aligned}|H|^2_{\omega\to 0}&\approx\frac{1}{1+\omega^{2N}}+1-2\sqrt{\frac{1}{1+\omega^{2N}}}\left[1-\frac{1}{2}\left(\frac{\omega^3}{3\sin\frac{3\pi}{2N}}\right)^2\right]\\&\approx\frac{1}{1+\omega^{2N}}+1-2\sqrt{\frac{1}{1+\omega^{2N}}}+\left(\frac{\omega^3}{3\sin\frac{3\pi}{2N}}\right)^2\\&\approx\left(1-\sqrt{\frac{1}{1+\omega^{2N}}}\right)^2+\left(\frac{\omega^3}{3\sin\frac{3\pi}{2N}}\right)^2.\end{aligned}$$

Since $\sqrt{1/(1+\omega^{2N})}$ approaches $1-(1/2)\omega^{2N}$ as $\omega\to 0$, we can write

$$|H|^2\approx\left(\frac{1}{2}\omega^{2N}\right)^2+\left(\frac{\omega^3}{3\sin\frac{3\pi}{2N}}\right)^2.$$

For $N>1$, the second term dominates as $\omega\to 0$, and $|H|$ is proportional to ω^3, for an asymptotic variation of 60 dB/decade, independent of the order N. However, for $N=1$, the first term dominates at low frequencies, and the asymptotic gain variation is only 40 dB/decade. The result for $N=1$ is to be expected, since the form of the filter for $N=1$ is the same as for the first-order Bessel-derived MDS filter.

These results are verified by the computer simulation of the Butterworth-derived MDS filter shown in Fig. 5. The simulation is carried out for Butterworth filters of orders 1, 2, 3, 4, and 6. It can be seen that for $N>1$, the rejection below the cutoff frequency is much sharper than with a Bessel filter, and that there is a much sharper "corner" in the response near the cutoff frequency. In fact, for the higher order filters, the response just below the cutoff frequency approaches 80 dB/decade. However, the good amplitude response near the −3-dB cutoff frequency is accompanied by a relatively poor phase response in this region, especially for high-order filters. Therefore, these filters would not be very useful unless the signal spectrum was kept at least about half an octave above the cutoff frequency of the filter. Thus, if the noise and signal were separated by an octave or more, the Butterworth-derived filter might yield a better noise rejection than the Bessel-derived filter. For example, it can be seen from Figs. 2 and 5 that a third-order Butterworth-derived MDS filter actually has a better phase response above its −3-dB cutoff frequency than a third- or fourth-order Bessel-derived filter has above its −3-dB cutoff frequency.

Other low-pass configurations such as a Chebycheff filter, or a transitional Bessel–Butterworth response, were not investigated but may be able to supply a 60-dB/decade low-frequency rejection with a better phase response just below the cutoff frequency than does a Butterworth-derived filter.

IV. Conclusions

In practical applications, the MDS high-pass filter may be a desirable alternative to a digital FIR linear phase filter if a digital processor of capability sufficient for the FIR algorithm is not already part of the system. In most such applications, the delay element will be the part of the MDS filter that will be the most expensive and produce the most noise and distortion. However, there are many frequency ranges for which a relatively inexpensive and high quality delay is available.

Though a Bessel-derived filter can provide a response sufficient for many purposes, with very little phase error, it can yield an asymptotic attenuation of only 40 dB/decade. Higher attenuations can be attained by using other types of low-pass filter in the MDS configuration, but there may be some attendant sacrifice in phase characteristic. Higher attenuations could also be attained by cascading sections each having a shorter delay, as two 5-ms sections each derived from a three-pole Bessel low pass filter instead of a 10-ms filter derived from a six-pole filter. By a proper selection of the cascaded sections it may be possible to partially cancel the phase errors caused by the lower order filters used, however the design of such multisection MDS filters is not considered further in this work.

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